Curvature and control of trajectories of second order ODEs

Bronisław Jakubczyk

The behaviour of nearby trajectories of a system of nonlinear equations

$$\ddot{x} = F(x, \dot{x}), \quad x(t) \in R^n, \qquad (SODE)$$

can be studied by using a $n \times n$ matrix $K(x)$ which is called the curvature or Jacobi endomorphism. For a trajectory $x^*(t)$ of (SODE), one can use the linearized along $x^*(t)$ equation which can be simplified to the equation $\ddot{\delta x} = K_t \delta x$ by a linear time-depending transformation, where $K_t = K(x^*(t))$.

We will present an explicit formula for the curvature matrix K and show how certain aspects of behavior of trajectories of (SODE) can be deduced from K. Such aspects include diverging property or presence of conjugate times (times after which perturbed trajectories return to the reference trajectory and, with an impuls control, can be placed back on it). As an example we will discuss the motion of a charged particle in the electromagnetic field (important e.g. in electron microscopy and ion trapping).

The curvature matrix was found as one of the invariants of (SODE) in the first half of XX century (Kosambi, Cartan and Chern). More recently it was used for analyzing the geodesics of Finsler structures.

Author: Bronisław Jakubczyk
Affiliation: *Institute of Mathematics, Polish Academy of Sciences*
Poland
e-mail: `b.jakubczyk@impan.pl`

Conjugate points and curvature in nonlinear control systems

B. Jakubczyk, W. Kryński

We will consider control-affine systems

$$\dot{x} = X(x) + \sum u_i Y_i(x) \qquad (CS)$$

on a phase space $M = R^n$ or a manifold, where the vector fields X and $Y_1, \ldots, Y_m$ are called the drift and control vector fields, respectively. Our motivating problem is to try to understand how the trajectories of the drift X change under small impulses of the controls u_i, i.e., under small changes of the velocity $\dot{x}$ produced by such impulses at certain times.

For this and other aims it is useful to consider the equivalence of systems (CS) defined by the invertible linear gauge transformations $u \mapsto Gu$, where u are elements of R^m. Then the system can be replaced by the pair (X, V), where V is the distribution spanned by $Y_1, \ldots, Y_m$. We will concentrate the discussion on the case where $n = 2m$ or $2m + 1$.

We will show that under natural regularity assumptions on the pair (X, V) one can define geometric objects (characteristics, invariants) like connection, curvature (Jacobi endomorphism), Jacobi equation and vector fields, analogous to those used in Riemannian geometry. Interesting enough, there is no metric given a priori. However, one can assign to (X, V) a class of metrics invariant under the connection. Using such geometric invariants one can analyse some of the aspects of behaviour of trajectories of the drift X, in particular existence and separation times of conjugate points. An example of gravitating bodies will be presented.

References

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First Author: Bronisław Jakubczyk
Affiliation: *Institute of Mathematics, Polish Academy of Sciences Poland*
e-mail: b.jakubczyk@impan.pl

Second Author: Wojciech Kryński
Affiliation: *Institute of Mathematics, Polish Academy of Sciences Poland*
e-mail: krynski@impan.pl