Global and local bifurcations of homoclinic solutions

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Bounded entire solutions are natural candidates for bifurcating objects in analysing parametrised nonautonomous evolutionary equations. However, appropriate explicit and sufficient conditions for bifurcations require combining recent functional, analytical methods from abstract bifurcation theory for Fredholm operators with tools from topological dynamics.

In this talk, we present how to study the following problem:

$$\begin{cases} \dot{x}(t) = f(t, x(t), \lambda) \\ \lim_{t \to \pm\infty} x(t) = 0, \end{cases} \tag{1}$$

where $x \in W_0^{1,\infty}(\mathbb{R}, \mathbb{R}^d)$, $f : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ is a Carathéodory function and $f(t, 0, \lambda) = 0$ for all $(t, \lambda) \in \mathbb{R} \times \mathbb{R}$. Since the last assumption implies that the pair $(0, \lambda)$ is a solution of (1), we are interested in nontrivial solutions, i.e., solutions (x, λ) with $x \neq 0$.

In particular, we present how to show the existence of the so-called bifurcation points for (1). Recall that a point $\lambda^* \in \mathbb{R}$ is a bifurcation point for homoclinic solutions of (1), if there is a sequence $(x_n, \lambda_n) \in W^{1,\infty}(\mathbb{R}, \mathbb{R}^d) \times \mathbb{R}$ such that $x_n \neq 0$ is a solution of (1) with $\lambda_n \to \lambda^*$ and $x_n \to 0$ in $W^{1,\infty}(\mathbb{R}, \mathbb{R}^d)$. Furthermore, we establish an alternative classifying the shape of global bifurcating branches of homoclinic solutions to Carathéodory differential equations.

Our approach uses the concept of parity (a crucial tool and topological invariant in the abstract bifurcation theory of nonlinear Fredholm operators, as developed by Fitzpatrick, Pejsachowicz and Rabier), the Evans function $E(\lambda)$, which was initially used in the stability analysis of travelling waves in evolutionary PDEs, and some fundamental methods from topological dynamics, such as the hull of a function and the Bebutov flow.

References

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